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ON THE MIXING PROPERTY FOR HYPERBOLIC SYSTEMS

BY

MARTINE BABILLOT

MAPMO, Université d'Orléans
Rue de Chartres, B.P. 6759, 45067 Orléans Cedex 2, France
email: mbab@labomath.univ-orleans.fr

ABSTRACT

We describe an elementary argument from abstract ergodic theory that can be used to prove mixing of hyperbolic flows. We use this argument to prove the mixing property of product measures for geodesic flows on (not necessarily compact) negatively curved manifolds. We also show the mixing property for the measure of maximal entropy of a compact rank-one manifold.

Introduction

The main purpose of this note is to present a simple argument to prove the strong mixing property for dynamical systems with some hyperbolic behavior. It is based on an abstract lemma from ergodic theory that shows, at the level of mixing, a symmetry between past and future. This appears as a counterpart for the equality between limits of Birkhoff sums for positive and negative times that is used in the celebrated Hopf argument for proving ergodicity of Anosov flows, or the equality between the Pinsker algebras of a system and its time reversal, which was used by Sinai to prove the much stronger K-property of hyperbolic systems, from which the mixing property is usually deduced.

Hence, this approach avoids the difficult identification of the Pinsker algebras, and thus allows one to deal with a number of systems for which it is still unclear whether the K-property holds. In particular, one may consider geodesic flows on not necessarily compact negatively curved manifolds and their ergodic properties

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with respect to Patterson–Sullivan measures, or geodesic flows on compact manifolds endowed with a non-positively curved metric of geometric rank one and the mixing property for Knieper's measure of maximal entropy. More singular spaces, such as CAT(-1)-spaces or Gromov hyperbolic spaces, may also be considered. Let us recall that proving the mixing property of some relevant measure, besides having its own interest from the ergodic theoretical point of view, gives a way towards counting lattice points in negatively curved manifolds and closed orbits of hyperbolic systems and also for proving equidistribution results, as was first shown by Margulis [M], and used since in different contexts; see, e.g., [EMcM], [L], [Ro].

This note is organised as follows. In section 1, we state and prove the main lemma. In section 2, we apply it to the systems mentioned above in order to show that, as for classical hyperbolic systems, the main obstruction to mixing is of a topological nature. In section 3, we connect the mixing property of the geodesic flow with an equidistribution property for horospheres.

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1. A lemma from ergodic theory

The following lemma is an application to ergodic theory of the semi-group properties discussed in [HMP] for the weak limits of characters. It allows us to give a more precise version of the well-known fact that, if the Fourier transform of a measure on $\mathbf{R}$ or $\mathbf{Z}$ does not vanish at $+\infty$, then it does not vanish at $-\infty$ either.

LEMMA 1: *Let $(X, \mathcal{B}, m, (T_t)_{t\in A})$ be a measure preserving dynamical system, where $(X, \mathcal{B})$ is a standard Borel space, m a (possibly unbounded) Borel measure on $(X, \mathcal{B})$ and $(T_t)_{t\in A}$ an action of a locally compact second countable abelian group A on X by measure preserving transformations. Let $\varphi \in L^2(X, m)$ be a real-valued function on X such that $\int \varphi \, dm = 0$ if m is finite.*

If there exists a sequence (t_n) going to infinity in A such that $\varphi \circ T_{t_n}$ does not converge to 0 in the weak-L^2 topology, then there exist a sequence (s_n) going to *infinity in A and a non-constant function ψ in $L^2(X,m)$ such that*

$$\varphi \circ T_{s_n} \to \psi \quad \text{and} \quad \varphi \circ T_{-s_n} \to \psi \quad \textit{in the weak-}L^2 \textit{ topology.}$$

Remark 1: We do not know whether this lemma is true for an action of a non-abelian group.

Remark 2: In the particular case of a **Z**-action, the Lemma suggests the following question: let E^+ (resp. E^-) be the set of those functions $\psi \in L^2(m)$ for which there exists $\phi \in L^2(m)$ and a sequence (n_k) tending to $+\infty$ (resp. $-\infty$) such that ψ is a weak limit of $\phi \circ T^{n_k}$. Does one have $E^+ = E^-$? After this paper was submitted, we learnt from Y. Derriennic and T. Downarowicz that the answer is yes and that it could be proved using similar ideas.

Proof of the Lemma: Let $\hat{A}$ be the dual group of A, that is the group of continuous characters of A into the unit circle S^1. By the spectral theorem, there exists an isometry between the closure in L^2 of the **C**-linear span of $\{\varphi \circ T_t, t \in A\}$ and $L^2(\hat{A}, \mu_\varphi, \mathbf{C})$ where the spectral measure μ_φ is the probability measure on $\hat{A}$ defined by the equality

$$\forall t \in A \quad \int_X \varphi \circ T_t \varphi dm = \int_{\hat{A}} F_t(x) d\mu_\varphi(x)$$

and where $F_t\colon \hat{A} \to S^1$ is the dual character on $\hat{A}$ given by $F_t(x) = x(t)$. This isometry maps φ to $F_0 \equiv 1$ and intertwines the composition by T_t with the multiplication by F_t. Since, by assumption, $\varphi \circ T_{t_n}$ does not converge to 0, we may choose a non-zero element $F \in L^2(\hat{A}, \mu_\varphi)$ such that, after passing to a subsequence, F_{t_n} converges to F in the weak-L^2 topology. But it is known that the set W of limit points of characters is closed under pointwise multiplication [HMP]. Let us recall briefly the argument. Let $F = \lim F_{t_n}$ and $F' = \lim F_{t'_m}$ be two such limit points, where both sequence (t_n) and (t'_m) go to the point at infinity in A. For any $f \in L^2(\hat{A}, \mu_\varphi, \mathbf{C})$, we have

$$\int F' F_{t_n} f d\mu_\varphi = \lim_{m \to +\infty} \int F_{t'_m} F_{t_n} f \, d\mu_\varphi$$

for any fixed n. Hence, $F' F_{t_n}$ belongs to W. Then, observing that $|F'| \leq 1$ so that $F' f \in L^2$, we also have

$$\int F F' f d\mu_\varphi = \lim_{n \to +\infty} \int F_{t_n} F' f d\mu_\varphi.$$

We get that FF' is a weak limit of a sequence in W, and thus also belongs to W since W is closed.

Now, since $F = \lim F_{t_n}$, the conjugate function $\bar{F}$ is the limit of $\bar{F}_{t_n} = F_{-t_n}$ and, by the above, the function $F\bar{F}$ is the limit of F_{s_n} for some sequence s_n going to infinity. Because $F\bar{F}$ is now real-valued, it is also the limit of the sequence F_{-s_n}. Coming back to $L^2(X, \mathcal{A}, m)$, this gives a non-zero limit point ψ of both sequences $\varphi \circ T_{s_n}$ and $\varphi \circ T_{-s_n}$. It remains to show that ψ is not constant. This is so in the infinite measure case, since ψ is in L^2. In the finite measure case, this follows from the fact that ψ is non-zero and satisfies $\int \psi dm = \lim \int \varphi \circ T_{s_n} dm = 0$. ∎

Lemma 1 will be used through the classical

FACT: *Let (φ_n) be a sequence converging weakly in $L^2(X, \mathcal{B}, m)$ to some function ψ. Then there is a subsequence (φ_{n_k}) such that the Cesaro averages*

$$A_{K^2} = \frac{1}{K^2} \sum_{k=1}^{K^2} \varphi_{n_k}$$

converge almost surely to ψ.

Indeed, it follows from the proof of the Banach–Saks theorem (see, e.g., [RN], p. 80) that there is a subsequence (φ_{n_k}) such that the square of the L^2-norm of $A_K - \psi$ is $O(1/K)$. The almost-sure convergence of (A_{K^2}) is then a simple consequence of the Borel–Cantelli lemma.

To illustrate the previous lemma, let us give a direct proof of the well known fact that a hyperbolic automorphism T of the torus is mixing with respect to the Lebesgue measure. Recall that, for each point p of the torus, the stable leaf and the unstable leaf of p are transversal at p, and the Lebesgue measure m is absolutely continuous with respect to the product of the Lebesgue measures on the leaves. If T were not mixing, there would exist a continuous function ϕ such that $\int \phi dm = 0$ and $\phi \circ T^n$ does not go to 0 in the weak L^2 topology when n tends to infinity. Applying Lemma 1 gives a non-constant function ψ which is the almost sure limit of Cesaro averages of ϕ, for both positive times and negative times. By the continuity of ϕ and the hyperbolicity of T, we get that ψ is constant on almost any stable leaf, and on almost any unstable leaf. Fubini's theorem then implies that ψ is constant almost surely, a contradiction.

This argument can clearly be extended to hyperbolic transformations, for measures which have a product structure with respect to both stable and unstable foliations. For hyperbolic flows, however, the flow direction is neutral, and one has to deal with the fact that the flow may not be mixing for obvious topological reasons. For instance, an Anosov flow on a connected compact manifold is topologically mixing if and only if it is not a (constant) suspension of an Anosov diffeomorphism, or equivalently, if the direct sum of both strong stable and strong unstable distributions is not integrable [An]. By Arnold's argument, topological mixing holds for geodesic flows on compact negatively curved manifolds [AA], but may not hold for geodesic flows on more singular spaces like trees.

To illustrate Lemma 1 in the case of flows, we shall concentrate in the next section on geodesic flows of manifolds which are non-compact or non-negatively curved and show how Lemma 1 still allows one to reduce the mixing property of product measures to a topological property.

2. Geodesic flows

General references for this section are [Ba1], [Ba2] and [E]. We consider a Hadamard manifold $\tilde{M}$, i.e., a simply connected Riemannian manifold with everywhere non-positive sectional curvatures. The boundary at infinity $\partial\tilde{M}$ of $\tilde{M}$ is the set of equivalent classes of geodesic rays, where two rays are **equivalent** if and only if they remain at bounded distance apart. We shall deviate slightly from the terminology of [Ba1] and say that two rays r_1, r_2: $[0,+\infty[\to \tilde{M}$ are **asymptotic** if $\mathrm{dist}(r_1(t), r_2(t))$ goes to 0 as t tends to $+\infty$. For a point x at infinity, the **Busemann cocycle centered at** x is defined for any two points p and q in $\tilde{M}$ by

$$B_x(p,q) = \lim_{r\to x} \mathrm{dist}(p,r) - \mathrm{dist}(q,r).$$

The **horosphere** centered at x and based at p is the level set

$$\{q \in \tilde{M}; B_x(p,q) = 0\}.$$

Let Γ be a torsion-free discrete group of isometries of $\tilde{M}$. The geodesic flow $(\tilde{g}_t)$ acts on the unit tangent bundle $T^1\tilde{M}$ and commutes with the action of Γ. The quotient flow (g_t) on $\Gamma\backslash T^1\tilde{M}$ is the geodesic flow of the manifold $M = \Gamma\backslash\tilde{M}$. Since we are going to study the mixing property of (g_t) with respect to "product" measures, we have to describe more precisely the product structure of $T^1\tilde{M}$. Let us first consider the case of negatively curved manifolds.

2.1. Geodesic flows on negatively curved Riemannian manifolds. We assume now the curvature to be bounded away from 0. Then two distinct points at infinity are connected by a unique geodesic, so that the space $\mathcal{G}$ of oriented bi-infinite geodesics on $\tilde{M}$ coincides with the set $\partial^2\tilde{M}$ of pairs of distinct points at infinity. The unit tangent bundle $T^1\tilde{M}$ fibers over $\partial^2\tilde{M}$ by mapping any vector v to the pair $\pi(v) := (v^-, v^+)$ of the end points of the geodesic supported by v. Hence, $T^1\tilde{M}$ can be identified with $\partial^2\tilde{M} \times \mathbf{R}$ and the geodesic flow $(\tilde{g}_t)$ of $\tilde{M}$ acts by translation on $\mathbf{R}$.

In strictly negative curvature, two equivalent geodesic rays are asymptotic. This fact translates into hyperbolicity of the geodesic flow: For any $v \in T^1\tilde{M}$, let $\mathcal{H}_v$ be the horosphere centered at v^+ and based at the foot point of v. Then the set $W^{ss}(v)$ of inward unit normal vectors to $\mathcal{H}_v$ is the strong stable leaf of the vector v, which means that for any $w \in W^{ss}(v)$ one has

$$\operatorname{dist}(\tilde{g}_t(v), \tilde{g}_t(w)) \to 0 \quad \text{as } t \to +\infty.$$

Accordingly, any two vectors in $W^{su}(v) := -W^{ss}(-v)$ get closer and closer when applying the geodesic flow $(\tilde{g}_t)$ with negative time $t \to -\infty$, so that $W^{su}(v)$ is the strong unstable leaf of v.

The interesting dynamics of the quotient flow (g_t) take place on the non-wandering set Ω of (g_t), which can be described as follows. Let first $\Lambda \subset \partial\tilde{M}$ be the **limit set** of Γ, that is the smallest closed Γ-invariant subset in the boundary at infinity. We assume that Γ is non-elementary and therefore that the limit set is infinite. Then the set $\Lambda^2 - \mathrm{Diag}$ of pairs of distinct limit points is closed and Γ-invariant in $\partial^2\tilde{M}$, and therefore $\pi^{-1}(\Lambda^2 - \mathrm{Diag})$ is now invariant under both Γ and the geodesic flow $\tilde{g}_t$. Then the non-wandering set of (g_t) is $\Omega = \Gamma\backslash\pi^{-1}(\Lambda^2 - \mathrm{Diag})$ [E].

We shall be interested in invariant measures of the geodesic flow that have some product structure with respect to this product decomposition of $T^1\tilde{M}$. First note that any invariant measure m of the geodesic flow of M lifts to a measure $\tilde{m}$ on $T^1\tilde{M}$ which is both Γ and $\tilde{g}_t$-invariant. Therefore, one may decompose $\tilde{m}$ as $\tilde{m} = \mu \otimes dt$ where μ is a Γ-invariant Radon measure on the space of oriented geodesics $\partial^2\tilde{M}$, i.e., a **(geodesic) current** of Γ. Obviously, m is supported on the non-wandering set if and only if the corresponding current is supported on Λ^2.

Definition [Ka2]*:* An invariant measure of the geodesic flow on a negatively curved manifold M is called a quasi-product measure if there exist two probability measures ν^- and ν^+ on the boundary at infinity such that the geodesic current associated with m is equivalent to $\nu^- \otimes \nu^+$.

Let us recall the main examples of quasi-product measures. For co-compact Γ, quasi-product measures have been extensively studied since the Gibbs measure of any Hölder potential is of this type. For any group Γ, the Liouville measure and the harmonic measure are quasi-product measures: in the first case, $\nu^- = \nu^+ = \lambda_p$ is the Lebesgue measure on the sphere at infinity identified with the unit sphere around some point p, and in the second case, $\nu^- = \nu^+ = \nu_p$ is the harmonic measure of the Brownian motion starting at p [Ka1]. In all these cases, these measures are of full support.

The construction of quasi-product measures supported on the non-wandering set Ω goes back to Sullivan in the constant curvature case: he noticed that, in the ball model of the hyperbolic n-space, the measure $d\nu(x)d\nu(y)/|x-y|^{2\delta}$ is a geodesic current for Γ if ν is a Patterson measure of the group Γ and δ its critical exponent [Su].

The previous examples are particular cases of the following construction [Ka1]: a family $(\nu_p)_{p\in\tilde{M}}$ of finite measures on $\partial\tilde{M}$ is called a Γ-**invariant density** if the measures ν_p all belong to the same measure class and satisfy $\gamma^*\nu_p = \nu_{\gamma.p}$ for any isometry γ in Γ. Then the measure $\nu_p \otimes \nu_p$ is quasi-invariant under the diagonal action of Γ on $\partial^2\tilde{M}$. If, for some (and hence for all) p, its Radon–Nikodym derivative is a coboundary

$$\frac{d\gamma^*\,\nu_p}{d\nu_p}(x)\frac{d\gamma^*\,\nu_p}{d\nu_p}(y) = \frac{f_p(\gamma.x,\gamma.y)}{f_p(x,y)}$$

for some measurable function f_p: $\partial^2\tilde{M} \to]0,+\infty[$, then the measure $\mu_p(dx,dy) = f_p^{-1}(x,y)d\nu_p(x)d\nu_p(y)$ on $\partial^2 M$ is a geodesic current and thus gives rise to a quasi-product measure for the geodesic flow.

For instance, one gets the analogue of the Patterson–Sullivan measure in the variable curvature case when (ν_p) is a density on the limit set which is δ-**conformal**, i.e.,

$$\frac{d\nu_q}{d\nu_p}(x) = e^{-\delta B_x(p,q)}$$

[Ka1]. For co-compact Γ, it is also the measure of maximal entropy, and coincides therefore with the Bowen–Margulis measure [Ha1]. In the non-cocompact case, this measure is finite under some geometric conditions on the group Γ [DOP].

We can now state:

THEOREM 1: *Let $\tilde{M}$ be a simply connected Riemannian manifold with all sectional curvatures bounded away from 0 on the negative side. Let Γ be a non-elementary discrete group of isometries of $\tilde{M}$. Then either any quasi-product measure for the geodesic flow on $\Gamma\backslash T^1\tilde{M}$ is mixing, or the length spectrum of the geodesic flow is contained in a discrete subgroup of* $\mathbf{R}$.

Recall that the **length spectrum** of (g_t) is the collection of the lengths of all closed geodesics in $\Gamma\backslash T^1\tilde{M}$. The group generated by the length spectrum is known to be dense in $\mathbf{R}$ in a large number of cases, e.g., for compact manifolds, for rank-one locally symmetric spaces, for surfaces of variable negative curvature and for manifolds whose fundamental group contains parabolic elements. This property is equivalent to topological mixing of the geodesic flow restricted to Ω. We refer to [D] and the references therein.

Note that we do not assume the measure to be finite. In the infinite measure case, by the mixing property, we mean the following: for any Borel sets A and B with finite measure, the measure of $A \cap g_t B$ goes to 0 as t tends to $+\infty$. Equivalently, $\varphi \circ g_t$ converges weakly to 0 for any function φ in a dense subspace of L^2. In this case, this property is indeed very weak: it neither implies conservativity nor ergodicity of the flow. Moreover, it is easily seen to hold when the flow is totally dissipative.

It seems that Theorem 1 was not known in that generality, but there were many special results, where the assertion could often be strengthened: for Gibbs measures in negatively curved compact manifolds, the geodesic flow is known to be Bernoulli. When $\tilde{M}$ is a rank-one symmetric space and Γ a lattice, the mixing property of the Liouville measure is a special case of the Howe–Moore theorem; when Γ is supposed to be geometrically finite, the Patterson–Sullivan measure is also Bernouilli [Ru], [N]. For real hyperbolic spaces, Roblin has studied bi-conformal infinite invariant measures (where the densities ν^- and ν^+ are conformal, but not necessarily with the same exponent), and has given the exact rate of decay towards 0 [Ro].

Proof of Theorem 1: Recall from [Ka2] that a conservative quasi-product measure m is supported on the non-wandering set Ω, and is ergodic. Let μ be the ergodic current associated with m. Suppose that m is not mixing. Then, there exists a continuous function φ on Ω with compact support such that $\varphi \circ g_t$ does not converge weakly to 0. By the results of Section 1, we may find a non-constant function ψ which is the almost sure limit of Cesaro averages of φ for positive and negative times. Let $\tilde{\psi}$ be its lift to the universal cover $T^1\tilde{M}$. We first smooth $\tilde{\psi}$ along the flow by considering the function (also denoted $\tilde{\psi}$) $v \to \int_0^\epsilon \tilde{\psi}(\tilde{g}_s(v))ds$. Choosing ϵ sufficiently small ensures that it is not constant, and now there exists a set E_0 of full μ-measure in $\partial^2 M$ such that, for each $v \in \pi^{-1}(E_0)$, the function $t \to \tilde{\psi}(\tilde{g}_t v)$ is well defined and continuous at any real value t. The closed subgroup of $\mathbf{R}$ given by the periods of $t \to \tilde{\psi}(\tilde{g}_t v)$ only depends of the geodesic (x, y) containing v, so that we get a measurable map from E_0 into the set of closed subgroups of $\mathbf{R}$. Since this map is obviously Γ-invariant, it has to be constant μ-almost surely by the ergodicity of μ. Suppose that this subgroup is $\mathbf{R}$. This means that $\tilde{\psi}$ does not depend on time, so that it defines a Γ-invariant function on Λ^2 – Diag. By the ergodicity of μ again, this function is constant almost-surely, in contradiction with the fact that ψ is not constant. Therefore, there exists $a \geq 0$ such that this closed subgroup equals $a\mathbf{Z}$ on a set $E_1 \subset E_0$ of full μ-measure.

We shall now see that the cross-ratio of any four points in the limit set belongs to $a\mathbf{Z}$.

Note that $\tilde{\psi}$ is the almost-sure limit of the corresponding smoothed Cesaro averages of $\tilde{\varphi}$, so that if $\tilde{\psi}^+$ and $\tilde{\psi}^-$ are respectively the upper limit of these averages for positive and negative times, the set

$$E = \{(x,y) \in E_1; \quad \tilde{\psi}^+(v) = \tilde{\psi}^-(v) = \tilde{\psi}(v) \quad \text{for any } v \in \pi^{-1}(x,y)\}$$

has full μ-measure. By the hyperbolicity of the geodesic flow and the uniform continuity of ϕ, $\tilde{\psi}^+$ is constant along any stable leaf, and $\tilde{\psi}^-$ is constant along any unstable leaf.

We now use the product structure of the geodesic current μ to define

$$E^- = \{x \in \Lambda; (x,y') \in E \ \nu^+(dy')\text{-a.s}\}$$

and

$$E^+ = \{y \in \Lambda; (x',y) \in E \ \nu^-(dx')\text{-a.s}\}.$$

By Fubini's theorem, we have $\nu^-(E^-) = \nu^+(E^+) = 1$ and therefore $E^- \times E^+$ has full μ-measure.

Fix (x,y) in $E \cap (E^- \times E^+)$. We choose (x',y') such that the geodesics (x',y), (x,y') and (x',y') also belong to E. Since (x,y) is in $E^- \times E^+$, such (x',y') have full μ-measure. Now, we start with a vector v on the geodesic (x,y) and define inductively $v_1 \in W^{ss}(v)$ on the geodesic (x',y), $v_2 \in W^{su}(v_1)$ on the geodesic (x',y') and $v_3 \in W^{ss}(v_2)$ on the geodesic (x,y'). Finally, we end up with a vector v_4 on the unstable leaf of v_3 and supported on the initial geodesic (x,y). Then $\tilde{\psi}(v) = \tilde{\psi}(v_1) = \cdots = \tilde{\psi}(v_4)$ since all functions $\tilde{\psi}$, $\tilde{\psi}^-$ and $\tilde{\psi}^+$ coincide on these four geodesics. On the other hand, $v_4 = \tilde{g}_\tau(v)$ where $\tau = B(x,x',y,y')$ is precisely the cross-ratio of these points [O]. Therefore $B(x,x',y,y')$ is a period of $t \to \tilde{\psi}(\tilde{g}_t v)$ and thus belongs to $a\mathbf{Z}$. Since the support of the geodesic current μ coincides with Λ^2, the assertion follows by the continuity of the cross-ratio.

To conclude, we recall that the length of a closed geodesic represented by a hyperbolic element γ in the group equals twice the absolute value of $B(x_\gamma^+, x_\gamma^-, y, \gamma.y)$ for any $y \in \partial\tilde{M}$, where x_γ^+ and x_γ^-, the fixed points of γ, belong to Λ [O]. We get therefore that if a quasi-product measure m is not mixing, then the length spectrum is contained in a discrete subgroup of $\mathbf{R}$. To obtain the reverse implication, we observe that a measure with support Ω cannot be mixing if the flow is not topologically mixing on Ω. This proves Theorem 1. ∎

Remark: Theorem 1 can be extended to geodesic flows on Cat(-1) spaces, using the appropriate notion of the cross-ratio as defined in [HP].

We shall now study compact manifolds with some vanishing curvatures.

2.2. GEODESIC FLOWS ON RANK ONE COMPACT MANIFOLDS. In this section, we come back with a Hadamard manifold $\tilde{M}$, and we assume now that $\tilde{M}$ admits a co-compact torsion-free discrete group of isometries Γ and that the manifold $M = \Gamma\backslash\tilde{M}$ is irreducible. Then, according to the rank rigidity theorem, M is either a locally symmetric space of higher rank, or is rank one in the sense of Ballmann [Ba2], [BS]. In the first case, the geodesic flow is not ergodic with respect to the Liouville measure; in the second case, ergodicity of the Liouville measure is still an open question. We shall study here another invariant measure of the geodesic flow on a rank one manifold which has been constructed by Knieper as the unique measure maximizing entropy. This measure is known to be ergodic [Kn] and we shall in fact show:

THEOREM 2: *Let M be a compact rank-one manifold, and m_{top} be the measure of maximal entropy for the geodesic flow. Then m_{top} is mixing.*

To prove this, we first recall the definition and some properties of rank-one compact manifolds [Ba2], [Kn]. The **rank** of a vector v in the unit tangent bundle $T^1\tilde{M}$ of a Hadamard manifold is the dimension of the space of parallel Jacobi fields along the geodesic supported by v. One says that $\tilde{M}$ is rank one if there exists a rank-one vector. This infinitesimal condition translates to a global one when the manifold $\tilde{M}$ admits a discrete group of isometries Γ satisfying Eberlein's duality condition and, in particular, when Γ is co-compact: if a geodesic is rank one, then it does not bound a totally geodesic embedded flat strip ([Ba2] Prop. IV.4.4.). We get a splitting of $T^1\tilde{M}$ into two Γ-invariant sets which are respectively the open set of rank-one vectors, the **regular set**, and its complement.

Let $\mathcal{G}$ be the space of oriented non-parametrised geodesics, provided with the topology of Hausdorff convergence on compact sets, and $\mathcal{R} \subset \mathcal{G}$ be the open subset of regular geodesics. The main difference between negatively curved manifolds and rank-one manifolds is the fact that the map

$$\begin{aligned} \mathcal{G} &\mapsto \partial^2\tilde{M} \\ c &\to (c(-\infty), c(+\infty)) \end{aligned}$$

is no longer a homeomorphism between $\mathcal{G}$ and $\partial^2\tilde{M}$. However, it induces a homeomorphism between $\mathcal{R} \subset \mathcal{G}$ and its image, according to the following lemma:

LEMMA 2 ([Ba2], Lemma III.3.1): *Let c be a rank-one geodesic in $\tilde{M}$, and $c(0)$ a point on c. For each $R > 0$ there exist neighborhoods U of $c(-\infty)$ and V of $c(+\infty)$ in $\tilde{M} \cup \partial\tilde{M}$ such that, for any $x \in U$ and $y \in V$, there is a unique rank-one geodesic connecting x and y. Moreover, this geodesic intersects the ball of radius R centered at $c(0)$.*

Therefore, we may identify $\mathcal{R}$ with an open subset of $\partial^2\tilde{M}$ and the regular set with $\mathcal{R} \times \mathbf{R}$.

We shall say that two regular geodesics $xy \simeq (x, y)$ and $x'y' \simeq (x', y')$ define a **quadrilateral** (x, x', y, y') if there exist regular geodesics connecting the pairs (x, y') and (x', y), respectively. By Lemma 2, the set $\mathcal{Q}$ of quadrilaterals is an open neighborhood of the diagonal in $\mathcal{R} \times \mathcal{R}$.

PROPOSITION-DEFINITION: *Let (x, x', y, y') be a quadrilateral, and (p_n), (p'_n), (q_n) and (q'_n) be sequences of points in $\tilde{M}$ converging to x, x', y and y', respectively. Then the limit of*

$$\mathcal{B}(p_n, p'_n, q_n, q'_n) := \operatorname{dist}(p_n, q_n) + \operatorname{dist}(p'_n, q'_n) - \operatorname{dist}(p_n, q'_n) - \operatorname{dist}(p'_n, q_n)$$

exists as n tends to infinity and does not depend on the choice of the converging sequences. This limit is called the cross-ratio of the quadrilateral and denoted by $\mathcal{B}(x, x', y, y')$.

Proof: The proof of this proposition is similar to that given in [O] in the negative curvature case, except that one has to deal with the fact that two geodesics rays defining the same point at infinity are not necessarily asymptotic. Let H_x, $H_{x'}$, H_y and $H_{y'}$ be four horospheres centered at x, x', y and y', respectively, and M_0 be the complementary set in $\tilde{M}$ of the union of the four corresponding horoballs. These horospheres can be chosen to be pairwise disjoint, so that the intersection of the four geodesics xy, $x'y'$, xy' and $x'y$ with M_0 consists of four geodesic segments with respective length d_1, d_2, d_3 and d_4. Set

$$\mathcal{B}(x, x', y, y') = d_1 + d_2 - d_3 - d_4.$$

Note that this number does not depend on the choice of the horospheres. Since we have regular geodesics, the geodesics p_nq_n, $p'_nq'_n$, $p_nq'_n$ and p'_nq_n converge respectively towards xy, $x'y'$, xy' and $x'y$ by Lemma 2. In particular, their intersection with M_0 contains four segments which converge respectively towards the previous ones. To prove the proposition, it suffices therefore to show that the contribution of the remaining parts of these geodesics goes to 0 as n tends to infinity. Let us focus on one horosphere. For n large enough, denote by a (resp. a_n, a', a'_n) the point where the geodesic xy (resp. p_nq_n, xy', $p_nq'_n$) leaves H_x. We have to show that

$$B_{p_n}(a_n, a'_n) = \operatorname{dist}(a_n, p_n) - \operatorname{dist}(a'_n, p_n) \to 0 \quad \text{as } n \to +\infty.$$

Since (p_n) converges to x, the Busemann cocycle $B_{p_n}(.,.)$ converges uniformly on compact sets to $B_x(.,.)$. Since $a_n \to a$ and $a'_n \to a'$, we get $0 = B_x(a, a') = \lim_n B_{p_n}(a_n, a'_n)$ as desired. ∎

Corollary: *The cross-ratio is a continuous function on $\mathcal{Q}$.*

The following property of the cross-ratio has been pointed out to us by J.-P. Otal:

Fact: *The cross-ratio of two intersecting geodesics defining a quadrilateral is strictly positive.*

Proof: Let xy and $x'y'$ be two regular geodesics intersecting at some point p, and choose p_n and q_n (resp. p'_n and q'_n) on the geodesic xy (resp. $x'y'$). We have

$$\mathcal{B}(p_n, p'_n, q_n, q'_n) = \mathcal{B}(p_n, p, p'_n, p) + \mathcal{B}(p, q_n, p, q'_n).$$

Moreover,

$$\mathcal{B}(p_n, p, p'_n, p) \geq \mathcal{B}(p_1, p, p'_1, p) \quad \text{and} \quad \mathcal{B}(q_n, p, q'_n, p) \geq \mathcal{B}(q_1, p, q'_1, p)$$

for n large enough by the triangular inequality. The two right-hand-side terms are strictly positive since the geodesic segments p_1q_1 and $p'_1q'_1$ make an angle. Therefore, from the definition of the cross-ratio, $\mathcal{B}(x, x', y, y') > 0$. ∎

Let us now prove Theorem 2. The measure m_{top} of maximal entropy gives full mass to the regular set and is ergodic. Therefore, we may consider the geodesic current μ associated to it, which is a Γ-invariant ergodic measure giving full measure to $\mathcal{R} \subset \partial^2\tilde{M}$. Moreover, μ is equivalent to the product $\nu \otimes \nu$ for some probability measure ν with full support $\partial\tilde{M}$ [Kn]. Suppose, as in the negative curvature case, that m_{top} is not mixing. The same construction gives a set $E \subset \mathcal{R}$ of full μ-measure, a real number a and a Γ-invariant function $\tilde{\psi}$ defined on the regular set such that, for each $(x, y) \in E$, $\tilde{\psi}$ induces a continuous function along the geodesic xy with discrete group of periods $a\mathbf{Z}$. Consider the subset $E_r \subset E$ of recurrent geodesics in E. It has full μ-measure, by Poincaré recurrence since m_{top} is finite. On the other hand, for a regular recurrent vector the set of outward unit vectors to the horosphere defined by v is indeed a stable leaf for the geodesic flow (Prop. 4.1, [Kn]). Thus, the same argument applies and shows that the cross-ratio of ν^4-almost any quadrilateral belongs to $a\mathbf{Z}$. This holds therefore for any quadrilateral by continuity. As a consequence, the cross-ratio of any quadrilateral sufficiently close to (x,x,y,y) vanishes, since $\mathcal{B}(x,x,y,y) = 0$. This now leads to a contradiction, since choosing a point p on the geodesic xy and a regular geodesic $x'y'$ passing through p and sufficiently close to xy gives a quadrilateral with strictly positive cross-ratio. Theorem 2 is proved. ∎

Theorem 2 raises the question whether one could adapt the Margulis approach to solve Katok's conjecture on the number of closed geodesics of a compact rank-one manifold [BK].

3. Equidistribution of horospheres

It is a well known fact that any closed horocycle on a Riemann surface of finite volume gets equidistributed on the surface when pushed by the geodesic flow [Sa]. In fact, the same holds with a piece of any (unstable) horocycle, and we shall now see that this property can be extended to more general manifolds. More precisely, let $M = \Gamma\backslash\tilde{M}$ be a negatively curved manifold, with non-wandering set Ω and non-arithmetic length spectrum. We consider a quasi-product measure m of the geodesic flow with the following properties:

(a) m is a finite measure supported on Ω ($m(\Omega) = 1$),

(b) the geodesic current μ associated with m can be written as

$$d\mu(x,y) = f^{-1}(x,y)d\nu^-(x)d\nu^+(y)$$

where f is a *continuous* function on $\partial^2 M$.

Since m is finite, it is conservative, and it is mixing by Theorem 1.

We fix an origin o in $\tilde{M}$. Then, denoting v_0 the base boint of a unit vector v, we can identify $T^1\tilde{M}$ with $\partial^2\tilde{M}\times\mathbf{R}$ using the Γ-equivariant map $v \to (v^-, v^+, r)$ where $r = B_{v^-}(o, v_0)$. With this parametrization, a point x_0 in the boundary and a subset J not containing x_0 give rise to a piece of an unstable leaf in $T^1\tilde{M}$ which is $\{x_0\}\times J\times\{0\}$.

THEOREM 3: *With the above notations, let J be a closed subset of the boundary with non-zero ν^+ measure, and x_0 a point of the limit set not belonging to J.*

Then for any bounded and uniformly continuous function ϕ: $T^1M \to \mathbf{R}$, *one has*

$$\frac{1}{c_J(x_0)} \int_J \tilde{\phi}(x_0, y, t) \frac{d\nu^+(y)}{f(x_0, y)} \to \int_\Omega \phi dm \quad \text{as } t \to +\infty.$$

Here, $\tilde{\phi}$ *is the* Γ*-invariant lift of* ϕ *and* $c_J(x_0)$ *is the normalisation constant* $\int_J d\nu^+(y)/f(x_0, y)$.

Theorem 3 shows that the piece of the unstable horosphere $\tilde{g}_t(\{x_0\} \times J \times \{0\})$ gets equidistributed according to m when projected on the unit tangent bundle of M. This generalizes the case of a finite volume Riemann surface since, in the disk model of the hyperbolic plane, the geodesic current associated with the Liouville measure can be written as $dx\, dy/|x-y|^2$, and the measure $dy/|x_0 - y|^2$ on the horocycle $\{x_0\} \times J \times \{0\}$ is the Lebesgue measure.

Theorem 3 applies for the case of groups with finite Patterson–Sullivan measure m_{PS}, and shows in particular that closed horospheres get equidistributed according to m_{PS}, when integration on the horosphere is taken with respect to the Patterson measure at infinity.

Theorem 3 is a straightforward consequence of the mixing property of the geodesic flow with respect to m. Indeed, we may first suppose that ϕ is positive, and that J is small enough that there exists a compact neighborhood I_0 of x_0 with $I_0 \cap J = \emptyset$ and $\eta_0 > 0$ such that $I_0 \times J \times [0, \eta_0]$ embeds in $\Gamma \backslash T^1\tilde{M}$. Then for any $\epsilon > 0$, one may choose a compact neighborhood $I \subset I_0$ of x_0 and $0 < \eta \leq \eta_0$ such that the following two properties holds:

(i) $1 - \epsilon \leq f(x_0, y)/f(x, y) \leq 1 + \epsilon$ for any $(x, y) \in I \times J$,

(ii) $|\tilde{\phi}(x, y, t + r) - \tilde{\phi}(x_0, y, t)| \leq \epsilon$ for any $(x, y) \in I \times J$, $0 \leq r \leq \eta$ and any $t > 0$.

The second property holds since choosing I and η sufficiently small ensures that, for $x \in I$ and $0 \leq r \leq \eta$, the vectors (x, y, r) and $(x_0, y, 0)$ are close, uniformly in $y \in J$. Moreover, since they both belong to the same (weak) stable leaf, flowing them by the geodesic flow does not increase their distance.

We note that $\nu^-(I) > 0$ since the support of ν^- is the limit set of Γ.

It follows from these two estimates that the integral

$$\frac{1}{c_J(x_0)} \int_J \tilde{\phi}(x_0, y, t) \frac{dv^+(y)}{f(x_0, y)}$$

differs from

$$\frac{1}{c_J(x_0)\eta\nu^-(I)} \int_{I \times J \times [0,\eta]} \tilde{\phi}(x, y, t + r) \frac{d\nu^-(x) d\nu^+(y)}{f(x, y)} dr$$

by at most $\epsilon(1+\frac{1+\epsilon}{1-\epsilon}|\phi|_\infty)$. By the mixing property of the geodesic flow, we may now choose t large enough that $\int_{I\times J\times[0,\eta]}\tilde{\phi}(x,y,t+r)\frac{d\nu^-(x)d\nu^+(y)}{f(x,y)}dr$ differs by $m(I\times J\times[0,\eta])\int_\Omega \phi dm$ from at most $\epsilon\eta\nu^-(I)$. The conclusion follows.

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